Computer and Decision Making

An International Journal

www.comdem.org
eISSN: 3008-1416

Bi-parametric Similarity Measures for Complex Hesitant Fuzzy Sets with Applications to Pattern Recognition and Medical Diagnosis

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ARTICLE INFO

Article history:
Received 10 January 2025
Received in revised form 8 February 2025
Accepted 12 February 2025
Available online 13 February 2025

Keywords:
Similarity and Distance Measures, Medical Diagnosis, Pattern recognition

ABSTRACT

The complex hesitant fuzzy sets which are the extension of hesitant fuzzy sets can incorporate Tamir's complex fuzzy environment which makes them more appropriate to model the uncertainty and vagueness involved in real-life problems. Some of the traditional similarity measures may not capture all the information contained in the hesitant fuzzy set since such measures are often simple in handling fuzzy information. This study aims to propose new similarity measures under the notion of complex hesitant fuzzy sets. Moreover, we have delivered similarity measures and weighted similarity measures for Tamir's complex fuzzy set, hesitant fuzzy set, and complex hesitant fuzzy set for a better and more accurate comparison. It also reveals the advantages of the proposed measure over existing methods through mathematical derivation and theoretical analysis. New similarity measures integrate the merits of CFS and HFS to form a powerful tool for comparing datasets with both complexity and hesitancy. The proposed measure utilizes Tamir's complex fuzzy environment to represent membership degree; it is thereby capable of managing data that are not well handled by current theories. Using theoretical discussions and real-world examples such as medical diagnosis and pattern recognition, we show that the CHFS-based similarity measure improves the accuracy of decision-making. This work not only enriches the theory of fuzzy sets but also provides solutions for solving problems containing complex and hesitant data.

1. Introduction

During the decision-making process, a limited set of choices is evaluated based on several factors, and alternatives are ranked depending on their perceived correctness to the individual. The individual has the responsibility of making a decision, taking into account all relevant criteria collectively. When employing this approach, the rating values for every possible solution consider unbiased information

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https://doi.org/10.59543/comdem.v2i.13493

from experts in addition to factually accurate information. Conversely, it is widely acknowledged that the information they provide is a clear kind. The concept of linguistic variable and its application to approximate reasoning is given by Zadeh [1]. The technique of multiple-attribute decision-making examines and organizes the range of options in a manner that ensures their reliability and precision for the decision-makers [2]. This strategy is employed when each of the criteria is simultaneously considered. In the context of this methodology, predictions regarding the evaluation of each alternative incorporate both empirical data and expert perspectives. However, it is commonly assumed that the information they provide is accurate or up-to-date. The presence of numerous multi-attribute decision-making (MADM) difficulties in real-world scenarios can be attributed to the inherent uncertainty of the framework. These issues encompass data that is ambiguous, absent, or of dubious kind.

The notion of fuzzy set (FS) has been examined as an approach to handle ambiguity in multi-attribute decision-making, which is considered an essential concept. To manage it, Zadeh [3] originated FSs in 1965 as a modification of traditional set theory. Distance measures (DMs) and similarity measures (SMs) both are effective techniques for assessing the level of discrimination between the pairs of FSs. Therefore, FSs deal with imprecise data, which is inherent in real-world problems including medical diagnosis [4] and pattern recognition [5]. Furthermore, the idea of the complex fuzzy set (CFS) has been introduced by Ramot et al. [6] and the range of this structure is a unit disc in the complex plan.

Though FSs describe memberships in a clear-cut manner, CFS uses complex numbers to offer a more refined representation of uncertainty [7]. Tamir et al. [8] contributed by providing a new interpretation of the compound membership grade in complex fuzzy sets (CFSs); which has received much attention in recent years. By introducing the use of complex numbers into the membership function, this new interpretation generalizes the concept of FSs. The capacity of CFS to carry out situations at once that include uncertainties of language and numbers is its main advantage. Additionally, it also helps to create more realistic models which is the main objective of machine learning and therefore it enables the analytical work and the process of decision-making in ambiguous or unclear situations more precise and accurate.

When it comes to making decisions about things, people can sometimes be hesitant, but the ideas that have been discussed up till now are unable to account for this particular circumstance. To overcome these constraints, Torra [9] presented the hesitant fuzzy set (HFS). The membership function of HFS is defined as several subsets of [0, 1]. Additionally, many real-life challenges have been resolved through the utilization of the HFS [10-11]. In decision-making subjects, Farhadinia and Herrera-Viedma [12] proposed an MCDM approach based on extended HFS with unknown weights data. Alcantud and Giarlotta [13] explored necessary and possible HFS and initiated a novel model for decision-making problems. New SMs on certain FSs are also defined by Wang [14]. Chen et al. [15] described the comparison of SM of fuzzy values. A new way of approximating fuzzy DM and SM among generalized fuzzy numbers was presented by Guha and Chakraborty [16]. Zhang and Fu [17] analysed the SMs on three kinds of FSs. Wang et al. [18] proposed a comparative study of similarity measures. Boran and Akay proposed bi-parametric similarity measures based on intuitionistic fuzzy sets (IFSs) and proposed their application to medical diagnosis [19]. A similarity function to determine the degree of similarity among IFSs has been indicated by Chen and Randyanto [20]. Pappis and Karacapilidis [21] established a comparative analysis of fuzzy value SM. Lee-Kwang et al. [22] introduced the similarity measurements among FSs. The DMs and SMs for HFSs were investigated by Xu and Xia [23]. Singha et al. [24] approach carried out the modified DM on HFSs and its relevance in multi-criteria decision-making problems. The novel idea by Li et al. [25] involved determining distance measures on HFSs by the believability value, and their application in decision-making. A novel distance and SMs on HFSs, as well as their applications in multiple criteria decision-making, were presented by Li et al. [26]. Rezaei and Rezaei [27] proposed new SMs for HFSs. Novel DMs and SMs for HFSs are proposed by Tang et al. [28] one of the simplest ways to define the features between two objects is to measure their distance and similarity. In the last couple of years, researchers have developed different SMs aiming at the importance of similarity metrics.

FSs are individual and multi-variable approaches, for example, HFSs normally work fine when the uncertainty is single-dimension, but they get challenged when more than one dimension is involved. To close this distance, it is proposed the introduction of a new construct: a complex hesitant fuzzy set (CHFS). With this framework, membership values are complex numbers. These complex degrees of uncertainty shed light on the underpinnings of the complex communication infrastructure that was not previously available. CHFS is a suitable method that allows us to yield single solutions even in scenarios that are highly detailed with multifaceted aspects. This provides a more comprehensive and utility-based approach than the predecessors can offer, as it handles situations with the presence of both uncertainty and insecurity dimensions. A fundamental principle within human cognition is that of similarity. Three categories are used to group the DM and SM: There are three kinds of distance and similarity measures: 1) the measure based on metrics, 2) the measure based on set theory, and 3) the measure based on interpreters. Researchers have put a lot of stress on the criteria of similarity between FSs because they play a critical role in the theory of FSs. Moreover, the Tamir CFS environment is more valuable and reliable than Ramot's CFS environment because it can cover more data and in Tamir's CFS environment, the range of membership functions belong to the unit square of the complex plan as compared to the Ramot CFS environment in which the range of membership function belong to unit disc in complex plane. It means that to handle the more advanced and complex information the idea of Tamir's CFS is more reliable and more advanced problems can be handled in this way in which two-dimensional data is involved. Based on these observations and the advantages of Tamir's CFS environment in this article we have proposed the idea of

* Similarity Measures for Complex Fuzzy Sets
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* Application of the proposed SMs in pattern recognition and medical diagnosis.
* The comparative analysis of the proposed work shows the advantages of the delivered approach.

The rest of the article is arranged as follows, in section 2 we have proposed some basic definitions that can help to define the further theory. Section 3 is about the basic SMs based on Tamir's CFS environment. Section 4 is about the idea of SMs based on HFS. In section 5, we have delivered the notion of SMs based on the notion of CHFSs. Section 6 is about the application algorithm and the utilization of the proposed theory for medical diagnosis and pattern recognition. In section 7 we have proposed the comparative analysis of the delivered approach to show the superiority of the introduced work. Section 8 is about the concluding remarks.

2. Preliminaries

In this section, we reviewed some fundamental notions, including FSs, CFSs, and HFSs. Additionally, in this paper, F(X) shows the collection of all FSs, while $Ꞁ_{\overline{\overline{\mathrm{P}}}}(\hat{ꞔ})$ denotes membership degree (MD) in interval $[0,1]$. Furthermore, X will denote the universal set.

Definition 1: [3] A FS $\overline{\overline{\mathrm{P}}}$ is of the form

$$\overline{\overline{\mathrm{P}}} = \left\{\left(\hat{ꞔ}, Ꞁ_{\overline{\overline{\mathrm{P}}}}(\hat{ꞔ})\ \right) | \hat{ꞔ} \epsilon \mathrm{X}\right\}$$

Where $\text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}})$ denotes the MD and it is restricted to the closed interval $[0,1]$. The pair $\overline{\overline{\mathrm{P}}} = \left(\hat{\mathfrak{c}}, \text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}})\right)$ is referred to as a fuzzy number (FN).

Definition 2: [8] A CFS $\overline{\overline{\mathrm{P}}}$ is a notion of the form

$$\overline{\overline{\mathrm{P}}} = \left\{\left(\hat{\mathfrak{c}}, \text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}})\right) | \hat{\mathfrak{c}} \epsilon \mathrm{X}\right\}$$

Where $\text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) = \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) + \iota\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}})$ is demonstrated the complex-valued MD and $\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}), \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) \in [0,1]$ and $\iota = \sqrt{-1}$. Further the pair $\overline{\overline{\mathrm{P}}} = \left(\hat{\mathfrak{c}}, \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) + \iota\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}})\right)$ is called a complex fuzzy number (CFN).

Definition 3: [9] If X is a fixed set then the HFS on X can be defined by a function that takes as input X and gives as its output as a subset of [0, 1]. Mathematical representation of HS is given by

$$\overline{\overline{\mathrm{P}}} = \left\{\left(\hat{\mathfrak{c}}, \text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) | \hat{\mathfrak{c}} \epsilon \mathrm{X}\right)\right\}$$

Where $\text{A}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}) \subset [0,1]$.

Definition 4: [23] Let $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ be two HFSs on the universal set X, $\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right)$ is a SM among $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ if it satisfied the following conditions

(1) $0 \leq \dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) \leq 1$

(2) $\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 1 \text{ iff } \overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$

(3) $\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = \dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}\right)$

Definition 5: [23] Let $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ be two HFSs on the universal set X, $\mathrm{Ɖ}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right)$ is called DM among $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ if it met the following conditions

(1) $0 \leq \mathrm{Ɖ}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) \leq 1$

(2) $\mathrm{Ɖ}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 0 \text{ iff } \overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$

(3) $\mathrm{Ɖ}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = \mathrm{Ɖ}\left(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}\right)$

Based on the discussion above, it is evident that $\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 1 - \mathrm{Ɖ}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right)$

3. Similarity Measures for Complex Fuzzy Sets

This section introduces new SMs for Tamir's CFS environment. These SMs were designed to improve CFSs thereby increasing the comfort with which they can be evaluated for accuracy and reliability.

Definition 6: Suppose that $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ are two CFSs in X where $\mathrm{X} = \{\hat{\mathfrak{c}}_1, \hat{\mathfrak{c}}_2, \ldots \hat{\mathfrak{c}}_n\}$

$$\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 1 - \left[\frac{1}{2n(t+1)^p} \sum_{i=1}^{n} \left\{\begin{array}{c} \left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{\mathfrak{c}}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{\mathfrak{c}}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{\mathfrak{c}}_i)\right)\right|^p \end{array}\right\}\right]^{\frac{1}{p}}$$

Where $t = 2,3,4, \dots$ $p = 1,2,3, \dots$ and $i = 1,2,3, \dots$

Here, there are two parameters, t indicates the degree of uncertainty and p is the ℓ_p Norm.

Theorem 1: Let $\dot{\check{\mathrm{S}}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}})$ are the SMs among two CFSs $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ inX. Then

(1) $0 \le \dot{\check{\mathrm{S}}}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) \le 1$

(2) $\dot{\check{\mathrm{S}}}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) = 1$ iff $\overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$

(3) $\dot{\check{\mathrm{S}}}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) = \dot{\check{\mathrm{S}}}_{\mathrm{CFS}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}})$

Proof of (1): $\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) \in [0,1]$ and $\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) \in [0,1]$ then,

$$\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right] \in [0,1]$$

This implies that for $\mathrm{i} = 1$

We have

$$\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_1) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_1)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_1) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_1)\right)\right] \in [0,1]$$

For i=2

$$\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_2) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_2)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_2) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_2)\right)\right] \in [0,1]$$

By performing this procedure, we acquire

$$\sum_{i=1}^{n}\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right] \in n[0,1]$$

$$0 \le \left[\sum_{i=1}^{n}\left(\begin{array}{l}\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p\end{array}\right)\right]^{\frac{1}{p}} \le (2n(t+1)^p)^{\frac{1}{p}}$$

$$0 \le \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\left(\begin{array}{l}\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p\end{array}\right)\right]^{\frac{1}{p}} \le 1$$

$$-1 \le -\left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\left(\begin{array}{l}\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p\end{array}\right)\right]^{\frac{1}{p}} \le 0$$

$$0 \le 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\left(\begin{array}{l}\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p\end{array}\right)\right]^{\frac{1}{p}} \le 1$$

$$\Rightarrow 0 \le \dot{\check{\mathrm{S}}}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) \le 1$$

Proof of (2): $\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\begin{pmatrix}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}}$

Now as $\overline{\overline{P}} = \overline{\overline{Q}}$ $\dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i) = \dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)$ for $i = 1,2, \dots, n$ and $\tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i) = \tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)$ for $i = 1,2, \dots, n$ then

$$\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{2n(t+1)^p}[|1-1| + \dots + |1-1|]\right]^{\frac{1}{p}}$$

$$\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - 0$$

$$\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1$$

Proof of (3): $\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\begin{pmatrix}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}}$

$$\Rightarrow 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\begin{pmatrix}\left|t\left(-\dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) + \dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(-\tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) + \tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\begin{pmatrix}\left|t\left(-\dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) - \dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(-\tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) - \tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1 - \left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\begin{pmatrix}\left|t\left(\dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) - \dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i) - \tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}} = \dot{\check{S}}_{CFS}(\overline{\overline{Q}},\overline{\overline{P}})$$

$$\dot{\check{S}}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = \dot{\check{S}}_{CFS}(\overline{\overline{Q}},\overline{\overline{P}})$$

Definition 7: Let $\overline{\overline{P}},\overline{\overline{Q}} \in \mathrm{CFS}(X)$, we define the WSMs as:

$$\dot{\check{S}}^{w}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i\begin{pmatrix}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \dot{\check{S}}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}_{\overline{\overline{P}}}(\hat{\varsigma}_i) - \tilde{U}_{\overline{\overline{Q}}}(\hat{\varsigma}_i)\right)\right|^p\end{pmatrix}\right]^{\frac{1}{p}}$$

Where $t = 2,3,4, \dots$, $p = 1,2,3, \dots$ and $i = 1,2,3, \dots$

Where w_i is the weight of types $(\hat{\varsigma}_i)$ $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Theorem 2: Let $\dot{\check{S}}^{w}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}})$ is the WSMs among two CFSs $\overline{\overline{P}}$ and $\overline{\overline{Q}}$ in X. Then

(1) $0 \leq \dot{\check{S}}^{w}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) \leq 1$

(2) $\dot{\check{S}}^{w}_{CFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1$ iff $\overline{\overline{P}} = \overline{\overline{Q}}$

(3) $\dot{\check{S}}^{w}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}) = \dot{\check{S}}^{w}_{\mathrm{CFS}}(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}})$

Proof of (1): $\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) \in [0,1]$ and $\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) \in [0,1]$ then,

$$\left[\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) + \left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right] \in [0,1]$$

This implies that for $\mathrm{i} = 1$

We have

$$\left[\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_1) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_1)\right) + \left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_1) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_1)\right)\right] \in [0,1]$$

For $\mathrm{i} = 2$

$$\left[\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_2) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_2)\right) + \left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_2) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_2)\right)\right] \in [0,1]$$

By performing this procedure, we acquire

$$\sum_{i=1}^{n}\left[\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right) + \left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right] \in n[0,1]$$

$$0 \le \left[\sum_{i=1}^{n} w_i \begin{pmatrix} \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p \end{pmatrix}\right]^{\frac{1}{p}} \le (2w_i(t+1)^p)^{\frac{1}{p}}$$

$$0 \le \left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \begin{pmatrix} \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p \end{pmatrix}\right]^{\frac{1}{p}} \le 1$$

$$-1 \le -\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \begin{pmatrix} \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p \end{pmatrix}\right]^{\frac{1}{p}} \le 0$$

$$0 \le 1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \begin{pmatrix} \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p \end{pmatrix}\right]^{\frac{1}{p}} \le 1$$

$$\Rightarrow 0 \le \dot{\check{S}}^{w}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}) \le 1$$

Proof of (2): $\dot{\check{S}}^{w}_{\mathrm{CFS}}(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}) = 1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \begin{pmatrix} \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) - \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^p \end{pmatrix}\right]^{\frac{1}{p}}$

Now as $\overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$ $\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) = \dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)$ for $i = 1,2,\ldots,n$ and $\tilde{\check{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i) = \tilde{\check{U}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)$ for $i = 1,2,\ldots,n$ and $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w_i} = 1$ then

$$\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{2(t+1)^{p}}\left[(1)|1-1|+(1)|1-1|+\cdots(1)|1-1|\right]\right]^{\frac{1}{p}}$$

$$\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-0$$

$$\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1$$

Proof of (3): $\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{2(t+1)^{p}}\sum_{i=1}^{n}w_i\begin{pmatrix}\left|t\left(\mathring{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)-\mathring{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^{p}+\\ \left|t\left(\tilde{U}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)-\tilde{U}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)\right)\right|^{p}\end{pmatrix}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^{p}}\sum_{i=1}^{n}w_i\begin{pmatrix}\left|t\left(-\mathring{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)+\mathring{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}+\\ \left|t\left(-\tilde{U}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)+\tilde{U}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}\end{pmatrix}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^{p}}\sum_{i=1}^{n}w_i\begin{pmatrix}\left|t\left(-\mathring{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)-\mathring{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}+\\ \left|t\left(-\tilde{U}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)-\tilde{U}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}\end{pmatrix}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^{p}}\sum_{i=1}^{n}w_i\begin{pmatrix}\left|t\left(\mathring{\check{S}}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)-\mathring{\check{S}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}+\\ \left|t\left(\tilde{U}_{\overline{\overline{\mathrm{Q}}}}(\hat{c}_i)-\tilde{U}_{\overline{\overline{\mathrm{P}}}}(\hat{c}_i)\right)\right|^{p}\end{pmatrix}\right]^{\frac{1}{p}}=\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}}\right)$$

$$\Rightarrow \mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=\mathring{\check{S}}^{w}_{\mathrm{CFS}}\left(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}}\right)$$

4. Similarity Measures for Hesitant Fuzzy Sets

In this section, we have delivered the notion of SMs based on the idea of HFSs. Discussing the data in the HFS environment decreases the chance of data loss because in this case, the MD is the subset of [0, 1] and more data can be covered. The overall discussion is given by

Definition 8: Suppose that $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ are two HFSs in X where $\mathrm{X}=\{\hat{c}_1,\hat{c}_2,\ldots\hat{c}_{\mathrm{n}}\}$

$$\mathring{\check{S}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\mathrm{l}}\sum_{\mathrm{j}=1}^{\mathrm{l}}\left[\left|t\left(\mathring{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i)-\mathring{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$$

Where $t=2,3,4,\ldots$ $p=1,2,3,\ldots$ and $\mathrm{i}=1,2,3,\ldots$

Here, there are two parameters, t indicates the degree of uncertainty and p is the l_p norm.

Theorem 3: Let $\mathring{\check{S}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)$ is the SMs among two HFSs $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ in X. Then

(1) $0\le\mathring{\check{S}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)\le 1$

(2) $\mathring{\check{S}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1$ iff $\overline{\overline{\mathrm{P}}}=\overline{\overline{\mathrm{Q}}}$

(3) $\mathring{\check{S}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=\mathring{\check{S}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}}\right)$

Proof of (1): $\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\in[0,1]$

This implies that for $\mathrm{i}=1$

We have

$$\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_1)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_1)\right)\right\}\in[0,1]$$

For i=2

$$\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_2)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_2)\right)\right\}\in[0,1]$$

By performing this procedure, we acquire

$$\left[\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\right\}\right]\in n[0,1]$$

$$0\leq\left[\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\right|^{p}\right\}\right]^{\frac{1}{p}}\leq(n(t+1)^{p})^{\frac{1}{p}}$$

$$0\leq\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\right|^{p}\right\}\right]^{\frac{1}{p}}\leq 1$$

$$-1\leq-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left[\left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}\leq 0$$

$$0\leq 1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\sum_{\text{j}=1}^{\text{ƚ}}\left[\left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{j}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{j}}}(\hat{ȼ}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}\leq 1$$

$$\Rightarrow 0\leq\dot{\check{S}}_{{}''\mathrm{H}\ddot{\overline{\mathrm{F}}}\mathrm{S}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)\leq 1$$

Proof of (2): $\dot{\check{S}}_{{}''\mathrm{H}\ddot{\overline{\mathrm{F}}}\mathrm{S}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ƚ}}\left[\begin{matrix}\left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{ȼ}_i)\right)\right|^{p}+\\ \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{ȼ}_i)\right)\right|^{p}+\cdots+\\ \left|t\left(\dot{\check{S}}_{\overline{\overline{\mathrm{P}}}_{\text{ƚ}}}(\hat{ȼ}_i)-\dot{\check{S}}_{\overline{\overline{\mathrm{Q}}}_{\text{ƚ}}}(\hat{ȼ}_i)\right)\right|^{p}\end{matrix}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\left[\frac{1}{n(t+1)^{p}}\left\{\frac{1}{\text{ꝉ}}\begin{bmatrix}\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_1)\right)\right|^{\mathrm{p}}+\\ \left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_1)\right)\right|^{\mathrm{p}}+\\ \ldots+\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_1)\right)\right|^{\mathrm{p}}\end{bmatrix}+\frac{1}{\text{ꝉ}}\begin{bmatrix}\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_2)\right)\right|^{\mathrm{p}}+\\ \left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_2)\right)\right|^{\mathrm{p}}+\\ \ldots+\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_2)\right)\right|^{\mathrm{p}}\end{bmatrix}\right.\right.$$

$$\left.\left.+\frac{1}{\text{ꝉ}}\begin{bmatrix}\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_n)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_n)\right)\right|^{\mathrm{p}}+\\ \left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_n)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_n)\right)\right|^{\mathrm{p}}+\\ \ldots+\left|\mathrm{t}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_n)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ꝉ}}}(\hat{\mathfrak{c}}_n)\right)\right|^{\mathrm{p}}\end{bmatrix}\right\}\right]^{\frac{1}{p}}$$

Now as $\overline{\overline{\mathrm{P}}}_{\text{ɉ}}=\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}$ $\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)=\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)$, for $\text{ɉ}=1,2,\ldots,n$ and $i=1,2,\ldots.,n$ and $\tilde{\text{Ư}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)=\tilde{\text{Ư}}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{\mathfrak{c}}_i)$ for $\text{ɉ}=1,2,\ldots,n$ and $i=1,2,\ldots,n$

$$\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{n(t+1)^{p}}\left\{\frac{1}{\text{ꝉ}}\left[(1)|1-1|+\cdots+(1)|1-1|\right]\right\}\right]^{\frac{1}{p}}$$

$$\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-0$$

$$\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1$$

Proof of (3): $\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ꝉ}}\sum_{j=1}^{\text{ꝉ}}\left[\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ꝉ}}\sum_{j=1}^{\text{ꝉ}}\left[\left|t\left(-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)+\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ꝉ}}\sum_{j=1}^{\text{ꝉ}}\left[\left|t\left(-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\text{ꝉ}}\sum_{j=1}^{\text{ꝉ}}\left[\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}=\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}}\right)$$

$$\Rightarrow \dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}\left(\overline{\overline{\mathrm{Q}}},\overline{\overline{\mathrm{P}}}\right)$$

Definition 9: Let $\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\in\ \mathrm{HFS}(\mathrm{X})$, we define the WSMs as:

$$\dot{\check{\mathrm{S}}}_{\mathrm{HFS}}^{w}\left(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\right)=1-\left[\frac{1}{(t+1)^{p}}\sum_{i=1}^{n}w_i\left\{\frac{1}{\text{ꝉ}}\sum_{\text{ɉ}=1}^{\text{ꝉ}}\left[\left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$$

Where $t=2,3,4,\ldots$ $p=1,2,3,\ldots$ and $i=1,2,3,\ldots$, w_i is the weight of types $(\hat{\mathfrak{c}}_{\mathrm{i}})$ $\mathrm{w_i}\in[0,1]$ and $\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{w_i}=1$.

Here, there are two parameters, t indicates the degree of uncertainty and p is the $ⱡ_p$ norm.

Theorem 4: Let $\check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}})$ is the WSMs among two HFSs $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ in X. Then

(1) $0 \le \check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) \le 1$

(2) $\check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) = 1$ iff $\overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$

(3) $\check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) = \check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}})$

Proof of (1): $\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right) \in [0,1]$ then

This implies that for $\mathrm{i} = 1$

We have

$$\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_1) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_1)\right)\right\} \in [0,1]$$

For i=2

$$\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_2) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_2)\right)\right\} \in [0,1]$$

By performing this procedure, we acquire

$$\left[\sum_{i=1}^{n}\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right)\right\}\right] \in n[0,1]$$

$$0 \le \left[\sum_{i=1}^{n}\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left|t\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right)\right|^{p}\right\}\right]^{\frac{1}{p}} \le (w_i(t+1)^p)^{\frac{1}{p}}$$

$$0 \le \left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left|t\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right)\right|^{p}\right\}\right]^{\frac{1}{p}} \le 1$$

$$-1 \le -\left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left[\left|t\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}} \le 0$$

$$0 \le 1 - \left[\frac{1}{n(t+1)^p}\sum_{i=1}^{n} w_i\left\{\frac{1}{ⱡ}\sum_{j=1}^{ⱡ}\left[\left|t\left(\check{S}_{\overline{\overline{\mathrm{P}}}_j}(\hat{c}_i) - \check{S}_{\overline{\overline{\mathrm{Q}}}_j}(\hat{c}_i)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}} \le 1$$

$$0 \le \check{S}^{w}_{\mathrm{HFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) \le 1$$

Proof of (2): $\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\text{ŀ}}\left[\begin{array}{c}\left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_i) - \mathring{\check{S}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_i)\right)\right|^p + \\ \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_i) - \mathring{\check{S}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_i)\right)\right|^p + \cdots + \\ \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_i) - \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1 - \left[\frac{1}{(t+1)^p} w_i \left\{\frac{1}{\text{ŀ}}\left[\begin{array}{c}\left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_1) - \mathring{\check{S}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_1)\right)\right|^p + \\ \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_1) - \mathring{\check{S}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_1)\right)\right|^p + \\ \ldots + \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_1) - \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_1)\right)\right|^p\end{array}\right] + \frac{1}{\text{ŀ}}\left[\begin{array}{c}\left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_2) - \mathring{\check{S}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_2)\right)\right|^p + \\ \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_2) - \mathring{\check{S}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_2)\right)\right|^p + \\ \ldots + \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_2) - \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_2)\right)\right|^p\end{array}\right]\right.\right.$$

$$\left.\left. + \frac{1}{\text{ŀ}}\left[\begin{array}{c}\left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_n) - \mathring{\check{S}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_n)\right)\right|^p + \\ \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_n) - \mathring{\check{S}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_n)\right)\right|^p + \\ \ldots + \left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_n) - \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ŀ}}}(\hat{\mathfrak{c}}_n)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

Now as $\overline{\overline{P}}_{\text{ɉ}} = \overline{\overline{Q}}_{\text{ɉ}}$ $\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i) = \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)$, for $\text{ɉ} = 1,2,\ldots,n$ and $i = 1,2,\ldots.,n$ and $\tilde{\text{Ư}}_{\overline{\overline{P}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i) = \tilde{\text{Ư}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)$ for $\text{ɉ} = 1,2,\ldots,n$ and $i = 1,2,\ldots,n$

$$\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{(t+1)^p}\left\{\frac{1}{\text{ŀ}}[(1)|1-1| + \cdots + (1)|1-1|]\right\}\right]^{\frac{1}{p}}$$

$$\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - 0$$

$$\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1$$

Proof of (3): $\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = 1 - \left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\text{ŀ}}\sum_{j=1}^{\text{ŀ}}\left[\left|t\left(\mathring{\check{S}}_{\overline{\overline{P}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i) - \mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^p\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1 - \left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\text{ŀ}}\sum_{j=1}^{\text{ŀ}}\left[\left|t\left(-\mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i) + \mathring{\check{S}}_{\overline{\overline{P}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^p\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1 - \left[\frac{1}{(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\text{ŀ}}\sum_{j=1}^{\text{ŀ}}\left[\left|t\left(\mathring{\check{S}}_{\overline{\overline{Q}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i) - \mathring{\check{S}}_{\overline{\overline{P}}_{\text{ɉ}}}(\hat{\mathfrak{c}}_i)\right)\right|^p\right]\right\}\right]^{\frac{1}{p}} = \mathring{\check{S}}^{w}_{HFS}(\overline{\overline{Q}},\overline{\overline{P}})$$

$$\mathring{\check{S}}^{w}_{HFS}(\overline{\overline{P}},\overline{\overline{Q}}) = \mathring{\check{S}}^{w}_{HFS}(\overline{\overline{Q}},\overline{\overline{P}})$$

5. Similarity Measures for Complex Hesitant Fuzzy Sets

The idea of CHFS have been developed by Albaity et al. [29]. As SMs are an inherent concept of people's perception. SMs have applications in several areas inclusive of pattern recognition and medical diagnosis. This section introduces new SMs for CHFSs. The basic definition of CHFS is given by

Definition 10: [29] A complex hesitant fuzzy set is of the procedure $\overline{\overline{\mathrm{P}}} = \left\{\left(\hat{c}, A_{\overline{\overline{\mathrm{P}}}}(\hat{c})\right) | \hat{c} \epsilon X\right\}$ where $A_{\overline{\overline{\mathrm{P}}}}(\hat{c})$ is demonstrated by the finite subset of complex-valued membership grades within the unit square in a complex plan. Where $A_{\overline{\overline{\mathrm{P}}}}(\hat{c}) = \left\{\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}) + \iota \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c})\right)\right\}$ and $\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}), \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}}(\hat{c}) \in [0,1]$ and $\iota = \sqrt{-1}$

Definition 11: Assume that $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ are two CHFSs in X where $\mathrm{X} = \{\hat{c}_1, \hat{c}_2, \dots \hat{c}_{\mathrm{n}}\}$

$$\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 1 - \left[\frac{1}{2n(t+1)^p} \sum_{i=1}^{n} \left\{\frac{1}{\flat} \sum_{\mathrm{j}=1}^{\flat} \left[\begin{array}{l} \left|t\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right)\right|^p \end{array}\right]\right\}\right]^{\frac{1}{p}}$$

Where $t = 2,3,4, \dots$ $p = 1,2,3, \dots$ and $i = 1,2,3, \dots$

Here, there are two parameters, t indicates the degree of uncertainty and p is the $\flat_p$ norm.

Theorem 5: Let $\dot{\check{\mathrm{S}}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right)$ is the SMs among two CHFSs $\overline{\overline{\mathrm{P}}}$ and $\overline{\overline{\mathrm{Q}}}$ in X. Then

(1) $0 \leq \dot{\check{\mathrm{S}}}_{\mathrm{CHFS}}(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}) \leq 1$

(2) $\dot{\check{\mathrm{S}}}_{\mathrm{CHFS}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = 1$ iff $\overline{\overline{\mathrm{P}}} = \overline{\overline{\mathrm{Q}}}$

(3) $\dot{\check{\mathrm{S}}}_{\mathrm{CHFS}}\left(\overline{\overline{\mathrm{P}}}, \overline{\overline{\mathrm{Q}}}\right) = \dot{\check{\mathrm{S}}}_{\mathrm{CHFS}}\left(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}\right)$

Proof of (1): $\frac{1}{\flat}\sum_{\mathrm{j}=1}^{\flat}\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right) \in [0,1]$ and $\frac{1}{\flat}\sum_{\mathrm{j}=1}^{\flat}\left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right) \in [0,1]$ then

$$\frac{1}{\flat}\sum_{\mathrm{j}=1}^{\flat}\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_i) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_i)\right)\right] \in [0,1]$$

This implies that for $\mathrm{i} = 1$

We have

$$\left\{\frac{1}{\flat}\sum_{\mathrm{j}=1}^{\flat}\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_1) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_1)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_1) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_1)\right)\right]\right\} \in [0,1]$$

For i=2

$$\left\{\frac{1}{\flat}\sum_{\mathrm{j}=1}^{\flat}\left[\left(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_2) - \dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_2)\right) + \left(\tilde{\mathrm{U}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{c}_2) - \tilde{\mathrm{U}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{c}_2)\right)\right]\right\} \in [0,1]$$

By performing this procedure, we acquire

$$\left[\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)+\left(\tilde{\breve{U}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right]\right\}\right]\in n[0,1]$$

$$0\le\left[\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}\le(2n(t+1)^{p})^{\frac{1}{p}}$$

$$0\le\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}\le 1$$

$$-1\le-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}\le 0$$

$$0\le 1-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_j}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_j}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}\le 1$$

$$\Rightarrow 0\le\dot{\check{S}}_{\mathrm{CHFS}}\left(\overline{\overline{P}},\overline{\overline{Q}}\right)\le 1$$

Proof of (2): $\dot{\check{S}}_{\mathrm{CHFS}}\left(\overline{\overline{P}},\overline{\overline{Q}}\right)=1-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\cdots+\\\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_\flat}(\hat{\mathfrak{c}}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_\flat}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\\+\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_1}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_1}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_2}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_2}(\hat{\mathfrak{c}}_i)\right)\right|^{p}+\cdots+\\\left|t\left(\tilde{\breve{U}}_{\overline{\overline{P}}_\flat}(\hat{\mathfrak{c}}_i)-\tilde{\breve{U}}_{\overline{\overline{Q}}_\flat}(\hat{\mathfrak{c}}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\left[\frac{1}{2n(t+1)^{p}}\left\{\frac{1}{ƚ}\left[\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_1}(\hat{c}_1)-\dot{\check{S}}_{\overline{\overline{Q}}_1}(\hat{c}_1)\right)\right|^{p}+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_2}(\hat{c}_1)-\dot{\check{S}}_{\overline{\overline{Q}}_2}(\hat{c}_1)\right)\right|^{p}+\cdots\right.\right.\right.$$

$$+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_ƚ}(\hat{c}_1)-\dot{\check{S}}_{\overline{\overline{Q}}_ƚ}(\hat{c}_1)\right)\right|^{p}+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_1}(\hat{c}_1)-\tilde{U}'_{\overline{\overline{Q}}_1}(\hat{c}_1)\right)\right|^{p}+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_2}(\hat{c}_1)-\tilde{U}'_{\overline{\overline{Q}}_2}(\hat{c}_1)\right)\right|^{p}$$

$$\left.+\cdots+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_ƚ}(\hat{c}_1)-\tilde{U}'_{\overline{\overline{Q}}_ƚ}(\hat{c}_1)\right)\right|^{p}\right]$$

$$+\frac{1}{ƚ}\left[\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_1}(\hat{c}_2)-\dot{\check{S}}_{\overline{\overline{Q}}_1}(\hat{c}_2)\right)\right|^{p}+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_2}(\hat{c}_2)-\dot{\check{S}}_{\overline{\overline{Q}}_2}(\hat{c}_2)\right)\right|^{p}+\cdots\right.$$

$$+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_ƚ}(\hat{c}_2)-\dot{\check{S}}_{\overline{\overline{Q}}_ƚ}(\hat{c}_2)\right)\right|^{p}+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_1}(\hat{c}_2)-\tilde{U}'_{\overline{\overline{Q}}_1}(\hat{c}_2)\right)\right|^{p}+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_2}(\hat{c}_2)-\tilde{U}'_{\overline{\overline{Q}}_2}(\hat{c}_2)\right)\right|^{p}$$

$$\left.+\cdots+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_ƚ}(\hat{c}_2)-\tilde{U}'_{\overline{\overline{Q}}_ƚ}(\hat{c}_2)\right)\right|^{p}\right]+\cdots$$

$$+\frac{1}{ƚ}\left[\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_1}(\hat{c}_n)-\dot{\check{S}}_{\overline{\overline{Q}}_1}(\hat{c}_n)\right)\right|^{p}+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_2}(\hat{c}_n)-\dot{\check{S}}_{\overline{\overline{Q}}_2}(\hat{c}_n)\right)\right|^{p}+\cdots\right.$$

$$+\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_ƚ}(\hat{c}_1)-\dot{\check{S}}_{\overline{\overline{Q}}_ƚ}(\hat{c}_1)\right)\right|^{p}+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_1}(\hat{c}_n)-\tilde{U}'_{\overline{\overline{Q}}_1}(\hat{c}_n)\right)\right|^{p}$$

$$\left.\left.\left.+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_2}(\hat{c}_n)-\tilde{U}'_{\overline{\overline{Q}}_2}(\hat{c}_n)\right)\right|^{p}+\cdots+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_ƚ}(\hat{c}_n)-\tilde{U}'_{\overline{\overline{Q}}_ƚ}(\hat{c}_n)\right)\right|^{p}\right]\right\}\right]^{\frac{1}{p}}$$

Now as $\overline{\overline{P}}_j=\overline{\overline{Q}}_j$ $\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)=\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)$, for $j=1,2,\dots,n$ and $i=1,2,\dots,n$ and $\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i)=\tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)$ for $j=1,2,\dots,n$ and $i=1,2,\dots.,n$ then

$$\dot{\check{S}}_{CHFS}(\overline{\overline{P}},\overline{\overline{Q}})=1-\left[\frac{1}{2n(t+1)^{p}}\left\{\frac{1}{ƚ}\left[|1-1|+|1-1|+\cdots|1-1|\right]\right\}\right]^{\frac{1}{p}}$$

$$\dot{\check{S}}_{CHFS}(\overline{\overline{P}},\overline{\overline{Q}})=1-0$$

$$\dot{\check{S}}_{CHFS}(\overline{\overline{P}},\overline{\overline{Q}})=1$$

Proof of (3): $\dot{\check{S}}_{CHFS}(\overline{\overline{P}},\overline{\overline{Q}})=1-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{ƚ}\sum_{j=1}^{ƚ}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^{p}\\+\left|t\left(\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i)-\tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{ƚ}\sum_{j=1}^{ƚ}\left[\begin{array}{c}\left|t\left(-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)+\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^{p}\\+\left|t\left(-\tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)+\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2n(t+1)^{p}}\sum_{i=1}^{n}\left\{\frac{1}{ƚ}\sum_{j=1}^{ƚ}\left[\begin{array}{c}\left|t\left(-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^{p}\\+\left|t\left(-\tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)-\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^p\\+\left|t\left(\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_i)-\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}}=\dot{\check{S}}_{\mathrm{CHFS}}(\overline{\overline{Q}},\overline{\overline{P}})$$

$$\Rightarrow \dot{\check{S}}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})=\dot{\check{S}}_{\mathrm{CHFS}}(\overline{\overline{Q}},\overline{\overline{P}})$$

Definition 13: Let $\overline{\overline{P}},\overline{\overline{Q}}\in \mathrm{CHFS(X)}$, we define the WSMs as:

$$\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})=1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n}w_i\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p+\\\left|t\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_i)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

Where $t=2,3,4,\ldots$ $p=1,2,3,\ldots$ and $i=1,2,3,\ldots$,where w_i is the weight of types $(\hat{c}_i)$ $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$.

Theorem 6: Let $\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})$ is WSMs among two CHFSs $\overline{\overline{P}}$ and $\overline{\overline{Q}}$ in X. Then

(1) $0\le\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})\le 1$

(2) $\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})=1$ iff $\overline{\overline{P}}=\overline{\overline{Q}}$

(3) $\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{P}},\overline{\overline{Q}})=\dot{\check{S}}^{w}_{\mathrm{CHFS}}(\overline{\overline{Q}},\overline{\overline{P}})$

Proof of (1): $\frac{1}{\flat}\sum_{j=1}^{\flat}\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\in[0,1]$ and $\frac{1}{\flat}\sum_{j=1}^{\flat}\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_i)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\in[0,1]$ then

$$\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)+\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_i)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right]\in[0,1]$$

This implies that for $\mathrm{i}=1$

We have

$$\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_1)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_1)\right)+\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_1)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_1)\right)\right]\right\}\in[0,1]$$

For i=2

$$\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_2)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_2)\right)+\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_2)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_2)\right)\right]\right\}\in[0,1]$$

By performing this procedure, we acquire

$$\left[\sum_{i=1}^{n}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i)-\dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)+\left(\tilde{U}_{\overline{\overline{P}}_j}(\hat{c}_i)-\tilde{U}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right]\right\}\right]\in n[0,1]$$

$$0 \leq \left[\sum_{i=1}^{n} w_i \left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}} \leq (2w_i(t+1)^p)^{\frac{1}{p}}$$

$$0 \leq \left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}} \leq 1$$

$$-1 \leq -\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}} \leq 0$$

$$0 \leq 1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_j}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_j}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_j}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}} \leq 1$$

$$\Rightarrow 0 \leq \dot{\check{S}}^{w}_{\mathrm{CHFS}}\left(\overline{\overline{P}}, \overline{\overline{Q}}\right) \leq 1$$

Proof of (2): $\dot{\check{S}}_{\mathrm{CHFS}}\left(\overline{\overline{P}}, \overline{\overline{Q}}\right) = 1 - \left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n} w_i \left\{\frac{1}{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_1}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_1}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_2}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_2}(\hat{c}_i)\right)\right|^p + \cdots + \\ \left|t\left(\dot{\check{S}}_{\overline{\overline{P}}_\flat}(\hat{c}_i) - \dot{\check{S}}_{\overline{\overline{Q}}_\flat}(\hat{c}_i)\right)\right|^p \\ + \left|t\left(\tilde{U}'_{\overline{\overline{P}}_1}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_1}(\hat{c}_i)\right)\right|^p + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_2}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_2}(\hat{c}_i)\right)\right|^p + \cdots + \\ \left|t\left(\tilde{U}'_{\overline{\overline{P}}_\flat}(\hat{c}_i) - \tilde{U}'_{\overline{\overline{Q}}_\flat}(\hat{c}_i)\right)\right|^p\end{array}\right]\right\}\right]^{\frac{1}{p}}$

$$\Rightarrow 1-\Bigg[\frac{1}{2(t+1)^p}\Bigg\{\frac{1}{\flat}w_1\Big[\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}+\cdots$$

$$+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_1)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_1)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}$$

$$+\cdots+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_1)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}\Big]$$

$$+\frac{1}{\flat}w_2\Big[\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}+\cdots$$

$$+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_2)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_2)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_2)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}$$

$$+\cdots+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_2)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_2)\Big)\Big|^{\mathrm{p}}\Big]+\cdots$$

$$+\frac{1}{\flat}w_n\Big[\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_n)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_n)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_n)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_n)\Big)\Big|^{\mathrm{p}}+\cdots$$

$$+\Big|\mathrm{t}\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_1)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_1)\Big)\Big|^{\mathrm{p}}+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_1}(\hat{\mathfrak{c}}_n)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_1}(\hat{\mathfrak{c}}_n)\Big)\Big|^{\mathrm{p}}$$

$$+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_2}(\hat{\mathfrak{c}}_n)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_2}(\hat{\mathfrak{c}}_n)\Big)\Big|^{\mathrm{p}}+\cdots+\Big|\mathrm{t}\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_\flat}(\hat{\mathfrak{c}}_n)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_\flat}(\hat{\mathfrak{c}}_n)\Big)\Big|^{\mathrm{p}}\Big]\Bigg\}\Bigg]^{\frac{1}{p}}$$

Now as $\overline{\overline{\mathrm{P}}}_{\mathrm{j}}=\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}$ $\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)=\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)$, for $\mathrm{j}=1,2,\dots,n$ and $i=1,2,\dots.,n$ and $\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)=\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_j}(\hat{\mathfrak{c}}_i)$ for $\mathrm{j}=1,2,\dots,n$ and $i=1,2,\dots,n$, where w_i is the weight of features $(\mathrm{X_i})$ $\mathrm{w_i}\in[0,1]$ and $\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{w_i}=1$ then

$$\dot{\check{\mathrm{S}}}^{w}_{\mathrm{CHFS}}\big(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\big)=1-\left[\frac{1}{2(t+1)^p}\left\{\frac{1}{\flat}[(1)|1-1|+(1)|1-1|+\cdots(1)|1-1|]\right\}\right]^{\frac{1}{p}}$$

$$\dot{\check{\mathrm{S}}}^{w}_{\mathrm{CHFS}}\big(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\big)=1-0$$

$$\dot{\check{\mathrm{S}}}^{w}_{\mathrm{CHFS}}\big(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\big)=1$$

Proof of (3): $$\dot{\check{\mathrm{S}}}^{w}_{\mathrm{CHFS}}\big(\overline{\overline{\mathrm{P}}},\overline{\overline{\mathrm{Q}}}\big)=1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n}w_i\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\Big|t\Big(\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\\+\Big|t\Big(\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n}w_i\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\Big|t\Big(-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)+\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\\+\Big|t\Big(-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)+\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^p}\sum_{i=1}^{n}w_i\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\Big|t\Big(-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)-\dot{\check{\mathrm{S}}}_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\\+\Big|t\Big(-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{Q}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)-\tilde{\mathrm{U}}'_{\overline{\overline{\mathrm{P}}}_{\mathrm{j}}}(\hat{\mathfrak{c}}_i)\Big)\Big|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}$$

$$\Rightarrow 1-\left[\frac{1}{2(t+1)^{p}}\sum_{i=1}^{n}w_{i}\left\{\frac{1}{\flat}\sum_{j=1}^{\flat}\left[\begin{array}{c}\left|t\left(\dot{\check{S}}_{\overline{\overline{Q}}_{j}}(\hat{\mathfrak{e}}_{i})-\dot{\check{S}}_{\overline{\overline{P}}_{j}}(\hat{\mathfrak{e}}_{i})\right)\right|^{p}\\+\left|t\left(\tilde{U}_{\overline{\overline{Q}}_{j}}(\hat{\mathfrak{e}}_{i})-\tilde{U}_{\overline{\overline{P}}_{j}}(\hat{\mathfrak{e}}_{i})\right)\right|^{p}\end{array}\right]\right\}\right]^{\frac{1}{p}}=\dot{\check{S}}_{\mathrm{CHFS}}^{w}\left(\overline{\overline{Q}},\overline{\overline{P}}\right)$$

$$\Rightarrow \dot{\check{S}}_{\mathrm{CHFS}}^{w}\left(\overline{\overline{P}},\overline{\overline{Q}}\right)=\dot{\check{S}}_{\mathrm{CHFS}}^{w}\left(\overline{\overline{Q}},\overline{\overline{P}}\right)$$

6. Applications

In this section, the suggested SMs for CHFSs are used for medical diagnosis and pattern recognition. We present the following example to illustrate this.

6.1. Application in Medical Diagnosis

The manifestations of various illnesses are different. The medical diagnosis involves the identification of symptoms that point to the sort of illness a victim has contracted. The several signs in a victim is a sign set and a set of illnesses can be represented by different illnesses.

Step 1: Define the set of symptoms X and the set of illness P. Each illness is signified as HFN.

Step 2: Calculate SMs among the victim's symptoms $\overline{\overline{Q}}$ and each illness $\overline{\overline{P}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$.

Step 3: Calculate the degree of SMs among the victim's symptoms $\overline{\overline{Q}}$ and each illness using proposed SMs according to the definition 11**.**

Step 4: Rank the illness according to the most similar to the victim's symptoms $\overline{\overline{Q}}$**.**

Step 5: Find the illness with the highest degree of SMs among the victim's symptoms $\overline{\overline{Q}}$.

Step 6: End

Example 1: Assume that a set of diagnosis

$$\overline{\overline{P}}=\left\{\overline{\overline{P}}_{1}(\text{pneumonia}),\overline{\overline{P}}_{2}(\text{Flu}),\overline{\overline{P}}_{3}(\text{Malaria}),\overline{\overline{P}}_{4}(\text{Diarrhea})\right\}$$

and the set of symptoms $\mathrm{X}=\{\hat{\mathfrak{e}}_{1}(\text{shallow breathing}),\hat{\mathfrak{e}}_{2}(\text{cough}),\hat{\mathfrak{e}}_{3}(\text{Fever}),\hat{\mathfrak{e}}_{4}(\text{Bloating})\}$

The victim's symptoms may manifest as CHFSs in the following ways:

$$\overline{\overline{Q}}=\begin{Bmatrix}(\hat{\mathfrak{e}}_{1},\{0.61+0.92\iota,0.77+0.81\iota\}),(\hat{\mathfrak{e}}_{2},\{0.43+0.52\iota,0.76+0.88\iota\}),\\(\hat{\mathfrak{e}}_{3},\{0.22+0.91\iota,0.56+0.67\iota\}),(\hat{\mathfrak{e}}_{4},\{0.72+0.48\iota,0.67+0.85\iota\})\end{Bmatrix}$$

The indications of every illness $\overline{\overline{P}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ are expressed as CHFSs as follows:

$$\overline{\overline{P}}_{1}(\text{pneumonia})=\begin{Bmatrix}(\hat{\mathfrak{e}}_{1},\{0.27+0.66\iota,0.52+0.91\iota\}),(\hat{\mathfrak{e}}_{2},\{0.57+0.88\iota,0.29+0.62\iota\})\\(\hat{\mathfrak{e}}_{3},\{0.66+0.82\iota,0.58+0.68\iota\}),(\hat{\mathfrak{e}}_{4},\{0.43+0.65\iota,0.33+0.41\iota\})\end{Bmatrix}$$

$$\overline{\overline{P}}_{2}(\text{Flu})=\begin{Bmatrix}(\hat{\mathfrak{e}}_{1},\{0.33+0.67\iota,0.12+0.27\iota\}),(\hat{\mathfrak{e}}_{2},\{0.72+0.89\iota,0.42+0.90\iota\}),\\(\hat{\mathfrak{e}}_{3},\{0.11+0.34\iota,0.25+0.99\iota\}),(\hat{\mathfrak{e}}_{4},\{0.49+0.78\iota,0.86+0.79\iota\})\end{Bmatrix}$$

$$\overline{\overline{P}}_{3}(\text{Malaria})=\begin{Bmatrix}(\hat{\mathfrak{e}}_{1},\{0.21+0.39\iota,0.27+0.87\iota\}),(\hat{\mathfrak{e}}_{2},\{0.41+0.44\iota,0.39+0.48\iota\}),\\(\hat{\mathfrak{e}}_{3},\{0.50+0.71\iota,0.33+0.69\iota\}),(\hat{\mathfrak{e}}_{4},\{0.42+0.61\iota,0.55+0.69\iota\})\end{Bmatrix}$$

$$\overline{\overline{\mathrm{P}}}_4(\text{Diarrhea}) = \begin{Bmatrix} (\hat{\mathfrak{e}}_1, \{0.72 + 0.79\iota, 0.40 + 0.56\iota\}), (\hat{\mathfrak{e}}_2, \{0.62 + 0.69\iota, 0.58 + 0.92\iota\}), \\ (\hat{\mathfrak{e}}_3, \{0.16 + 0.31\iota, 0.44 + 0.66\iota\}), (\hat{\mathfrak{e}}_4, \{0.12 + 0.56\iota, 0.76 + 0.45\iota\}) \end{Bmatrix}$$

We need to determine which diseases, among $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1, 2, 3, 4)$ the victim $\overline{\overline{\mathrm{Q}}}$ has. To do this, we calculated the proposed SMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1, 2, 3, 4,)$ and summarized results in Table 1. As exposed in Table 1, the highest degree of similarity is between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_3$ as measured by proposed SMs. According to the proposed SMs, indicating that the victim $\overline{\overline{\mathrm{Q}}}$ is affected by malaria.The ranking of the proposed SMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1, 2, 3, 4,)$ is also shown in Table 1. A graphical representation of the proposed SMs can be seen in Figure 1.

Consideration of the weight of the fundamentals is crucial when dealing with actual decision-making issues. If we assume that the weight of every component $\mathrm{X}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ is 0.2, 0.8, 0.1 and 0.2, correspondingly. Thereupon the proposed WSMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ Values are given in Table 2. We need to find out which of the $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ diseases the victim $\overline{\overline{\mathrm{Q}}}$ has. For this, we calculated the proposed WSMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ and presented in Table 2. Table 2 makes this clear that the degree of WSMs among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathbf{P}}}_{\mathbf{3}}$ is maximum as measured by the proposed WSMs. This means that malaria disease has affected the victim $\overline{\overline{\mathrm{Q}}}$ has. The rank of the proposed WSMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ is shown in Table 2. The graphical demonstration of the proposed WSMs among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{e}}}(\hat{\mathfrak{e}} = 1,2,3,4)$ are established in Figure 2.

Table 1: Procedures for proposed SMs for medical diagnosis

SMs	$\dot{\check{\mathrm{S}}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_1)$	$\dot{\check{\mathrm{S}}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_2)$	$\dot{\check{\mathrm{S}}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_3)$	$\dot{\check{\mathrm{S}}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_4)$	Ranking
$\dot{\check{\mathrm{S}}}(\overline{\overline{\mathbf{Q}}}, \overline{\overline{\mathbf{P}}}_{\hat{\mathfrak{e}}})$	0.7954	0.7689	0.8235	$\mathbf{0.8075}$	$\overline{\overline{\mathbf{P}}}_{\mathbf{3}} > \overline{\overline{\mathbf{P}}}_{\mathbf{4}} > \overline{\overline{\mathbf{P}}}_{\mathbf{1}} > \overline{\overline{\mathbf{P}}}_{\mathbf{2}}$

Table 2: Procedures for proposed WSMs for medical diagnosis

WSMs	$\dot{\check{\mathrm{S}}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_1)$	$\dot{\check{\mathrm{S}}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_2)$	$\dot{\check{\mathrm{S}}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_3)$	$\dot{\check{\mathrm{S}}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_4)$	Ranking
$\dot{\check{\mathrm{S}}}_w(\overline{\overline{\mathbf{Q}}}, \overline{\overline{\mathbf{P}}}_{\hat{\mathfrak{e}}})$	0.9240	0.9143	0.9488	$\mathbf{0.9387}$	$\overline{\overline{\mathbf{P}}}_{\mathbf{3}} > \overline{\overline{\mathbf{P}}}_{\mathbf{4}} > \overline{\overline{\mathbf{P}}}_{\mathbf{1}} > \overline{\overline{\mathbf{P}}}_{\mathbf{2}}$

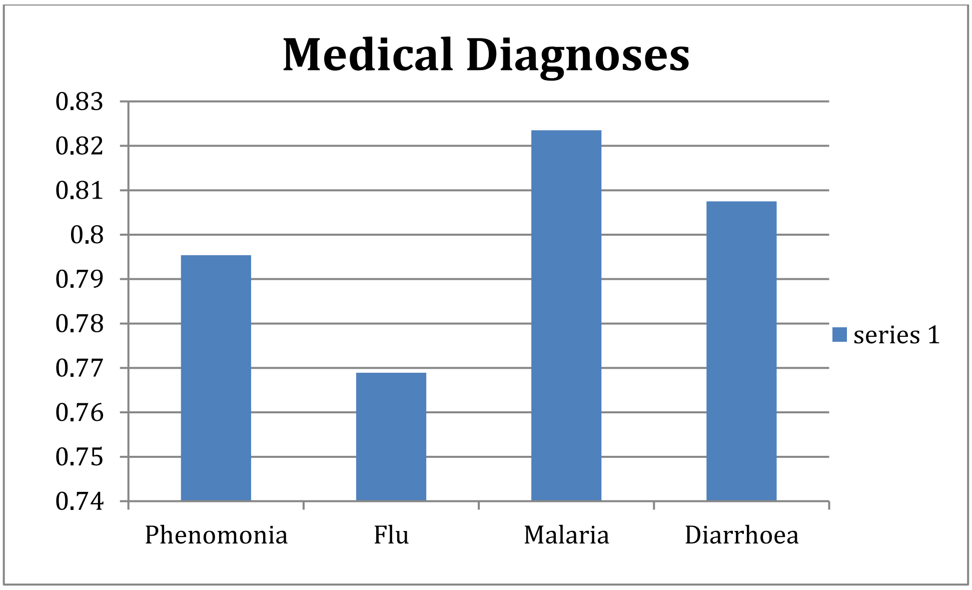

Figure 1. The graphical picture of proposed SMs for Medical diagnoses

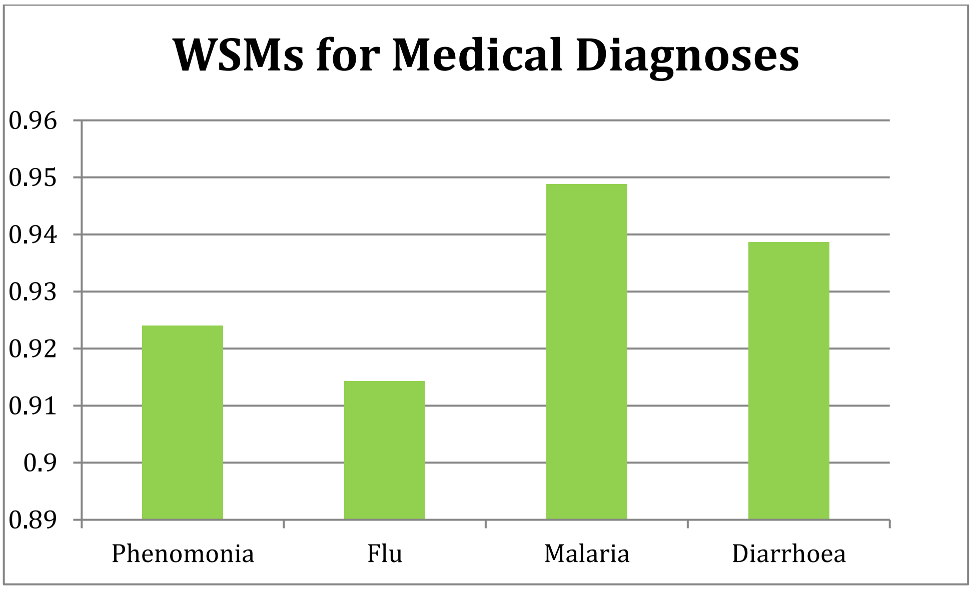

Figure 2.The graphical picture of proposed WSMs for Medical diagnoses

6.2. Application in Pattern Recognition

The pattern is everything, starting from the digital realm to this digital world. Either way should be watched. This can happen either through visual representation or with the use of mathematical tools like algorithms. The process of interpreting meaningful relationships and recurring patterns is Pattern Recognition (building PR).The algorithm feature of the machine learning which is responsible for the identification process. One of the best ways to present pattern recognition (building PR) is the labeling of data dependent on what previously is known about it or its statistics, derived from patterns or their depictions. On the other hand, pattern recognition (building PR) management serves as an essential component of application potential.

Step 1: Define the set of specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ and unspecified building substance $\overline{\overline{\mathrm{Q}}}$.

Step 2: Calculate SMs among specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ and unspecified building substance $\overline{\overline{\mathrm{Q}}}$.

Step 3: Calculate the degree of SMs among the specified building substance and unspecified building substance $\overline{\overline{\mathrm{Q}}}$ using proposed SMs according to the definition 11.

Step 4: Rank of the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ is most similar to the specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$.

Step 5: Find the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ with the highest similarity degree between specified building substances.

Step 6: End

Example 2: Assume there are four recognized building materials

$\overline{\overline{\mathrm{P}}}_1(\text{concrete}), \overline{\overline{\mathrm{P}}}_2(\text{paints}), \overline{\overline{\mathrm{P}}}_3(\text{steel}), \overline{\overline{\mathrm{P}}}_4(\text{brick})$ which are characterized by CHFSs

$$\overline{\overline{\mathrm{P}}}_1(\text{concrete}) = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.3 + 0.4\iota, 0.2 + 0.8\iota\}), (\hat{\mathfrak{c}}_2, \{0.8 + 0.5\iota, 0.9 + 0.6\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.1 + 0.7\iota, 0.4 + 0.7\iota\}), (\hat{\mathfrak{c}}_4, \{0.2 + 0.4\iota, 0.6 + 0.3\iota\}) \end{Bmatrix}$$

$$\overline{\overline{\mathrm{P}}}_2(\text{paints}) = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.4 + 0.6\iota, 0.6 + 0.7\iota\}), (\hat{\mathfrak{c}}_2, \{0.31 + 0.98\iota, 0.57 + 0.66\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.73 + 0.47\iota, 0.23 + 0.62\iota\}), (\hat{\mathfrak{c}}_4, \{0.41 + 0.68\iota, 0.39 + 0.68\iota\}) \end{Bmatrix}$$

$$\overline{\overline{\mathrm{P}}}_3\ (\text{steel}) = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.51 + 0.97\iota, 0.33 + 0.62\iota\}), (\hat{\mathfrak{c}}_2, \{0.23 + 0.11\iota, 0.51 + 0.73\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.67 + 0.43\iota, 0.82 + 0.84\iota\}), (\hat{\mathfrak{c}}_4, \{0.50 + 0.61\iota, 0.82 + 0.53\iota\}) \end{Bmatrix}$$

$$\overline{\overline{\mathrm{P}}}_4\ (\text{brick}) = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.72 + 0.79\iota, 0.40 + 0.56\iota\}), (\hat{\mathfrak{c}}_2, \{0.62 + 0.69\iota, 0.58 + 0.92\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.16 + 0.31\iota, 0.44 + 0.66\iota\}), (\hat{\mathfrak{c}}_4, \{0.15 + 0.91\iota, 0.33 + 0.63\iota\}) \end{Bmatrix}$$

Now, let it be an unspecified building substance that has to be recognized.

$$\overline{\overline{\mathrm{Q}}} = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.81 + 0.41\iota, 0.66 + 0.72\iota\}), (\hat{\mathfrak{c}}_2, \{0.46 + 0.65\iota, 0.91 + 0.93\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.81 + 0.48\iota, 0.32 + 0.47\iota\}), (\hat{\mathfrak{c}}_4, \{0.34 + 0.85\iota, 0.11 + 0.70\iota\}) \end{Bmatrix}$$

Where $\hat{\mathfrak{c}}_1 = \text{qurtz and fledspar}, \hat{\mathfrak{c}}_2 = \text{ironore}, \hat{\mathfrak{c}}_3 = \text{pigments}, \hat{\mathfrak{c}}_4 = \text{cement}$. For a clearer comprehension, let us assume $\hat{\mathfrak{c}}_1 = \text{qurtz and fledspar}$ in $\overline{\overline{\mathrm{P}}}_4$ similarly $\hat{\mathfrak{c}}_2 = \text{ironore}$ in $\overline{\overline{\mathrm{P}}}_3$, $\hat{\mathfrak{c}}_3 = \text{pigments}$ in $\overline{\overline{\mathrm{P}}}_2$ and $\hat{\mathfrak{c}}_4 = \text{cement}$ in $\overline{\overline{\mathrm{P}}}_1$. And as we have to go further in finding that the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belongs to which of the specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$. To this end, we obtained the values of the suggested SMs for $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ as in Table 3. As recorded in Table 3, we observed that SM among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_2$ is the greatest of all similarities among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$. This means for object of the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belongs to the specified building substance $\overline{\overline{\mathrm{P}}}_2$ .The ranking of suggested SMs among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ are also depicted in Table 3. The following Figure 3 illustrates the suggested SMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}$ that articulate the graphical representation for $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$.

The weight of components has significant importance in supporting real decision-making difficulties. Uncertainty we assume that the weight of each component. $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ is 0.2, 0.8, 0.1, and 0.3 correspondingly. Thereupon the proposed WSMs for $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ Values are given in Table 4. And as we have to go further in finding that the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belong to which of the specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3, \dots)$.To this end, we obtained the values of the proposed WSMs for $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4,)$ as in Table 4. As we recorded from Table 4, We observe that WSMs among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_2$ is greatest among all similarities between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$. This means that for the object of the unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belongs to the specified building substance $\overline{\overline{\mathrm{P}}}_2$.Ranking of proposed WSMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ are also depicted in Table 4. The following Figure 4 illustrates the proposed WSMs between $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}$ that articulate the graphical representation for $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$.

Table 3: Procedures for proposed SMs for building pattern recognition

SMs	$\mathring{\check{S}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_1)$	$\mathring{\check{S}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_2)$	$\mathring{\check{S}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_3)$	$\mathring{\check{S}}(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_4)$	Ranking
$\mathring{\check{S}}(\overline{\overline{\mathbf{Q}}}, \overline{\overline{\mathbf{P}}}_{\hat{\mathfrak{c}}})$	0.8324	0.8971	0.8266	$\mathbf{0.8765}$	$\overline{\overline{\mathbf{P}}}_2 > \overline{\overline{\mathbf{P}}}_4 > \overline{\overline{\mathbf{P}}}_1 > \overline{\overline{\mathbf{P}}}_3$

Table 4: Procedures for proposed WSMs for building pattern recognition

WSMs	$\mathring{\check{S}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_1)$	$\mathring{\check{S}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_2)$	$\mathring{\check{S}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_3)$	$\mathring{\check{S}}_w(\overline{\overline{\mathrm{Q}}}, \overline{\overline{\mathrm{P}}}_4)$	Ranking
$\mathring{\check{S}}_w(\overline{\overline{\mathbf{Q}}}, \overline{\overline{\mathbf{P}}}_{\hat{\mathfrak{c}}})$	0.9215	0.9478	0.9074	$\mathbf{0.9467}$	$\overline{\overline{\mathbf{P}}}_2 > \overline{\overline{\mathbf{P}}}_4 > \overline{\overline{\mathbf{P}}}_1 > \overline{\overline{\mathbf{P}}}_3$

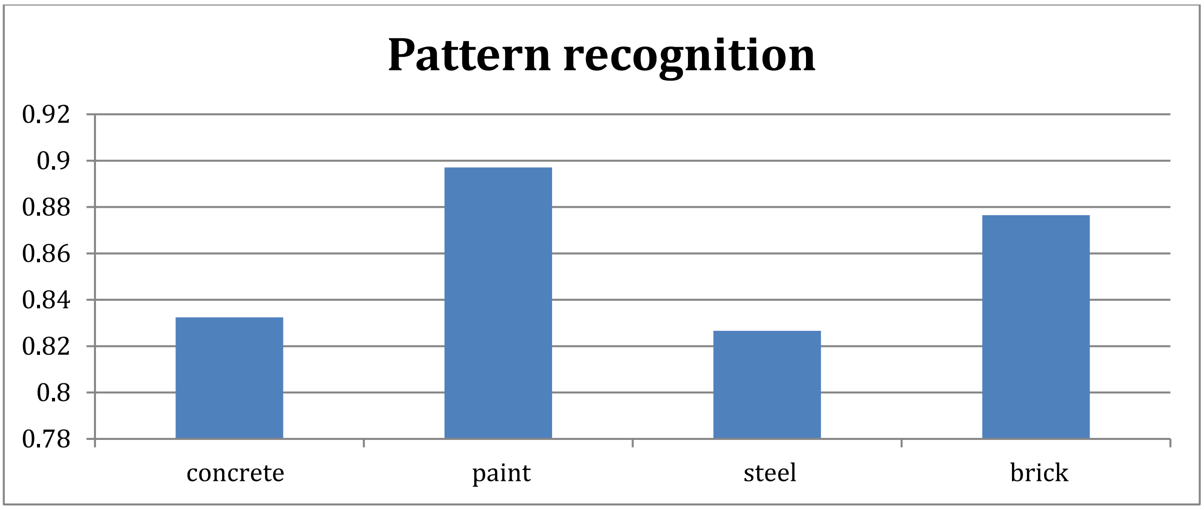

Figure 3.The graphical picture of proposed SMs for building pattern recognition

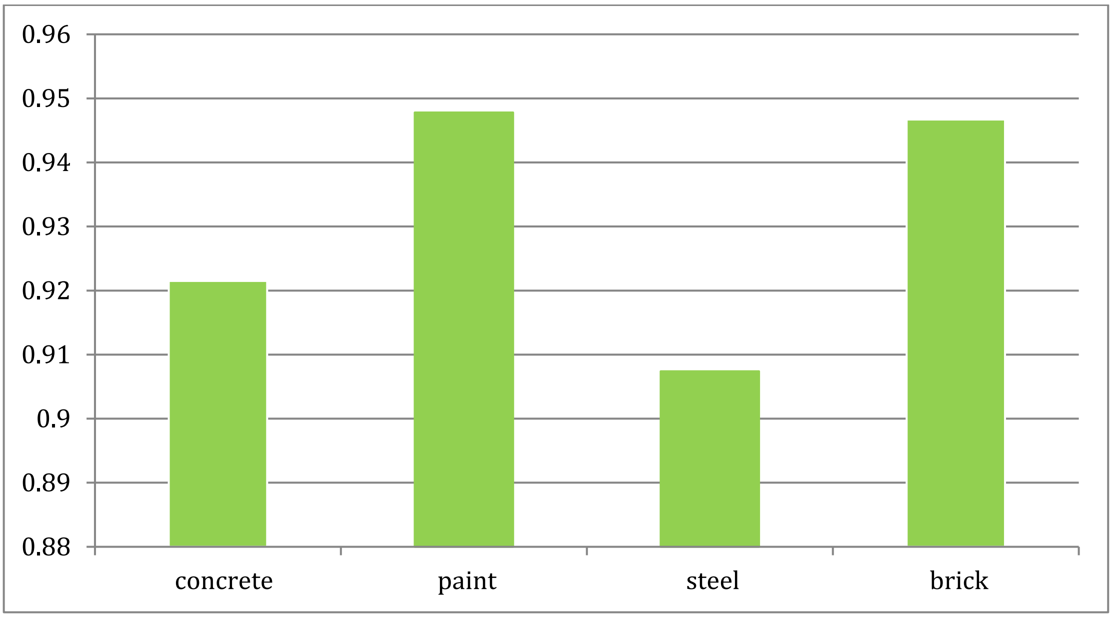

Figure 4. The graphical picture of proposed WSMs for building Pattern recognition

7. Comparison:

The goal of this section is to describe the comparative analysis of the suggested SMs introduced for Tamir's CFS environment, HFS and CHFSs to a few other SMs to demonstrate its efficacy.

Example 3. As mentioned, let four specified building substances be $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ which are characterized in the configuration of HFSs as discussed below

$$\overline{\overline{\mathrm{P}}}_1 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.54,0.72\}), (\hat{\mathfrak{c}}_2, \{0.47,0.81\}), \\ (\hat{\mathfrak{c}}_3, \{0.67,0.82\}), (\hat{\mathfrak{c}}_4, \{0.98,0.66\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_2 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.22, 0.18\}), (\hat{\mathfrak{c}}_2, \{0.91, 0.34\}), \\ (\hat{\mathfrak{c}}_3, \{0.58, 0.85\}), (\hat{\mathfrak{c}}_4, \{0.89, 0.17\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_3 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.37, 0.42\}), (\hat{\mathfrak{c}}_2, \{0.43, 0.62\}), \\ (\hat{\mathfrak{c}}_3, \{0.86, 0.89\}), (\hat{\mathfrak{c}}_4, \{0.51, 0.44\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_4 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.14, 0.19\}), (\hat{\mathfrak{c}}_2, \{0.71, 0.82\}), \\ (\hat{\mathfrak{c}}_3, \{0.63, 0.68\}), (\hat{\mathfrak{c}}_4, \{0.21, 0.33\}) \end{Bmatrix}$$

Thus, enables an unspecified building substance that wants to be specified.

$$\bar{\bar{Q}} = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.31, 0.62\}), (\hat{\mathfrak{c}}_2, \{0.56, 0.91\}), \\ (\hat{\mathfrak{c}}_3, \{0.34, 0.11\}), (\hat{\mathfrak{c}}_4, \{0.52, 0.69\}) \end{Bmatrix}$$

When we added 0ι formerly the data was obtainable in the form of HFSs changed into the CHFSs

$$\bar{\bar{P}}_1 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.54 + 0\iota, 0.72 + 0\iota\}), (\hat{\mathfrak{c}}_2, \{0.47 + 0\iota, 0.81 + 0\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.67 + 0\iota, 0.82 + 0\iota\}), (\hat{\mathfrak{c}}_4, \{0.98 + 0\iota, 0.66 + 0\iota\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_2 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.22 + 0\iota, 0.18 + 0\iota\}), (\hat{\mathfrak{c}}_2, \{0.91 + 0\iota, 0.34 + 0\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.58 + 0\iota, 0.85 + 0\iota\}), (\hat{\mathfrak{c}}_4, \{0.89 + 0\iota, 0.17 + 0\iota\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_3 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.37 + 0\iota, 0.42 + 0\iota\}), (\hat{\mathfrak{c}}_2, \{0.43 + 0\iota, 0.62 + 0\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.86 + 0\iota, 0.89 + 0\iota\}), (\hat{\mathfrak{c}}_4, \{0.51 + 0\iota, 0.44 + 0\iota\}) \end{Bmatrix}$$

$$\bar{\bar{P}}_4 = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.14 + 0\iota, 0.19 + 0\iota\}), (\hat{\mathfrak{c}}_2, \{0.71 + 0\iota, 0.82 + 0\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.63 + 0\iota, 0.68 + 0\iota\}), (\hat{\mathfrak{c}}_4, \{0.21 + 0\iota, 0.33 + 0\iota\}) \end{Bmatrix}$$

And

$$\bar{\bar{Q}} = \begin{Bmatrix} (\hat{\mathfrak{c}}_1, \{0.31 + 0\iota, 0.62 + 0\iota\}), (\hat{\mathfrak{c}}_2, \{0.56 + 0\iota, 0.91 + 0\iota\}), \\ (\hat{\mathfrak{c}}_3, \{0.34 + 0\iota, 0.11 + 0\iota\}), (\hat{\mathfrak{c}}_4, \{0.52 + 0\iota, 0.69 + 0\iota\}) \end{Bmatrix}$$

Table 5: Comparison of proposed SMs with some existing SMs

Methods	Score values	Ranking
Xu and Xia[23]	$\check{S}_{ne}(\bar{\bar{Q}}, \bar{\bar{P}}_1) = \mathbf{0.5237}$, $\check{S}_{ne}(\bar{\bar{Q}}, \bar{\bar{P}}_2) = \mathbf{0.3553}$, $\check{S}_{ne}(\bar{\bar{Q}}, \bar{\bar{P}}_3) = 0.4792$, $= \check{S}_{ne}(\bar{\bar{Q}}, \bar{\bar{P}}_4) = 0.5313$	$\bar{\bar{P}}_4 > \bar{\bar{P}}_1 > \bar{\bar{P}}_3 > \bar{\bar{P}}_2$

Rezaei at al[27]	$\dot{\check{S}}_{nh}(\overline{\overline{Q}}, \overline{\overline{P}}_1) = \mathbf{0.4875}$, $\dot{\check{S}}_{nh}(\overline{\overline{Q}}, \overline{\overline{P}}_2) = \mathbf{0.17}$ $\dot{\check{S}}_{nh}(\overline{\overline{Q}}, \overline{\overline{P}}_3) = 0.44$, $\dot{\check{S}}_{nh}(\overline{\overline{Q}}, \overline{\overline{P}}_4) = 0.4075$	$\overline{\overline{P}}_1 > \overline{\overline{P}}_3 > \overline{\overline{P}}_4 > \overline{\overline{P}}_2$
Li at al[26]	$\dot{\check{S}}_{g}(\overline{\overline{Q}}, \overline{\overline{P}}_1) = \mathbf{0.6635}$, $\dot{\check{S}}_{g}(\overline{\overline{Q}}, \overline{\overline{P}}_2) = \mathbf{0.5444}$, $\dot{\check{S}}_{g}(\overline{\overline{Q}}, \overline{\overline{P}}_3) = 0.6316$ $\dot{\check{S}}_{g}(Q, \overline{\overline{P}}_4) = 0.6685$	$\overline{\overline{P}}_4 > \overline{\overline{P}}_1 > \overline{\overline{P}}_3 > \overline{\overline{P}}_2$
Proposed SMs	$\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_1) = \mathbf{0.8351}$, $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_2) = \mathbf{0.8389}$ $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_3) = 0.8698$, $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_4) = 0.8828$	$\overline{\overline{P}}_4 > \overline{\overline{P}}_3 > \overline{\overline{P}}_2 > \overline{\overline{P}}_1$

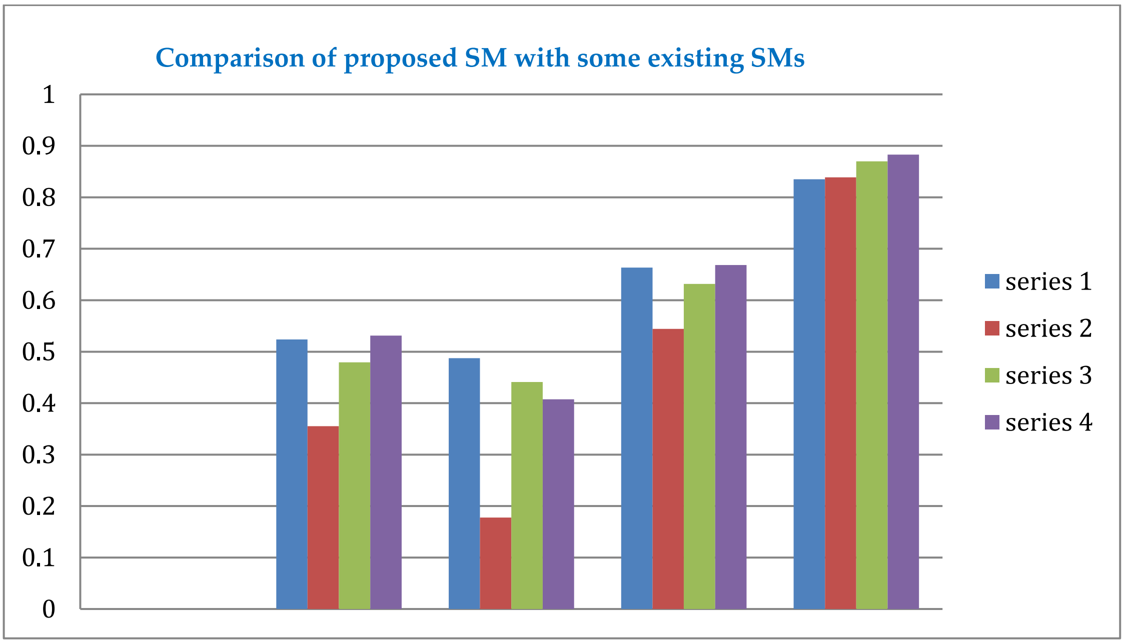

Figure 5 Geometrical representation of data given in Table 5

Table 6: Comparison of proposed SMs with some existing SMs

Methods	**Score values**	**Ranking**
Xu and Xia [23]	Failed	Failed
Rezaei at al. [27]	Failed	Failed
Li at al. [26]	Failed	Failed
Proposed SMs	$\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_1) = 0.8324$ $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_2) = 0.8971$ $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_3) = 0.8266$, $\dot{\check{S}}(\overline{\overline{Q}}, \overline{\overline{P}}_4) = 0.8765$	$\overline{\overline{P}}_2 > \overline{\overline{P}}_4 > \overline{\overline{P}}_1 > \overline{\overline{P}}_3$

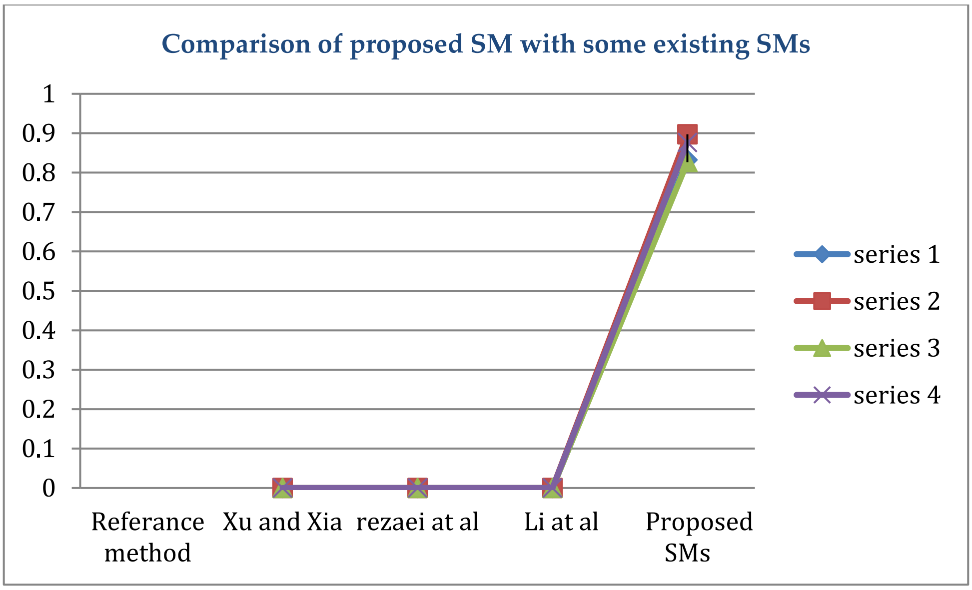

Figure 6: The observation among proposed SMs and existing SMs of Example 2.

For Example 3, we want to discover that the unknown building substance $\overline{\overline{\mathrm{Q}}}$ belongs to which of the specified building substance $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$. In Example 3, the data is in the shape of HFS, this type of data is solvable through the existing SMs as exposed in Table 5. By adding 0ι the data of Example 3, converted to the shape of CHFSs and through the proposed SMs, we can find the SMs among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ which are also prearranged in Table 5. Our proposed SMs exposed that unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belongs to the building substance $\overline{\overline{\mathrm{P}}}_4$ because the similarity among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_4$ is maximum. The rank of the proposed and current SMs is also defined in Table 5. Adjacent we have a graphical observation of the proposed and current SMs, which is denoted in Figure 5. Next, we deliberate the observation among interpreted and existing SMs for Example 2. In Example 2, the data are in the form of CHFSs. We recognize that not any SMs occur in the literature to explain this type of data. But through proposed SMs, we can discover the similarities among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$.From Table 6, we note that the data designated in Example 2, are resolvable by proposed SMs. The proposed SMs develop the similarity among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_{\hat{\mathfrak{c}}}(\hat{\mathfrak{c}} = 1,2,3,4)$ as demonstrated in Table 6. Our proposed SMs expose that unspecified building substance $\overline{\overline{\mathrm{Q}}}$ belongs to the specified building substance $\overline{\overline{\mathrm{P}}}_2$, because similarities among $\overline{\overline{\mathrm{Q}}}$ and $\overline{\overline{\mathrm{P}}}_2$ is maximum. The rank of the SMs is also presented in Table 6. Adjacent we have a graphical demonstration of observation of proposed and existing SMs which is represented in Figure 6.

As of the discussion overhead, it is clear that our explored SMs can place more fuzzy data and categorize it mainly in situations in actual life problems. In the interpretation of CHFS, we discussed the SMs; Our SMs are better suited for real-life problem-solving and the existing as well as our SMs have a broader coverage than the existing SMs.

8. Conclusion

In this article, we have explored new SMs for CFSs, HFSs, and CHFSs. We have developed a generalized approach to CHFSs, incorporating two key factors "t" which indicates the level of uncertainty, and "p" representing the "ℓ_p" norm. The proposed measures are more intuitive and simpler to apply across various contexts. We supported our recommendations with examples to demonstrate comparative effectiveness. The findings indicate that the suggested SMs can be utilized in future studies in fields such as pattern recognition, medical diagnosis, and data mining. Overall, the CHFS measures developed in this article are indeed more logical and comprehensible.

In the future, we can extend these notions to some aggregation theory proposed in [30-33]. We can explore the entropy measures for the proposed structure as given in [34-36].

Author Contributions
"Conceptualization, T.M. and J.A.; methodology, M.R.A.; software, M.H.F.; validation, U.R., J.A. and T.M.; formal analysis, T.M.; investigation, J.A. and M.H.F; resources, U.R.; data curation, M.H.F.; writing—original draft preparation, M.R.A.: visualization, J.A.; supervision, T.M.
Conflict of interest: About the publication of this manuscript, the authors declare that they have no conflict of interest.

Data availability: The data generated and analyzed for this study is contained entirely in the manuscript.

Ethics Declaration Statement: The authors state that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

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